Zoom Class 24/4/2020

Curvature at the Origin (By Newtonian Method)

We have $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

If the curve is $y=f(x)$ and is passing through origin $(0,0)$ then $x=0$ and $y=0$

$\therefore$ Let $y_{1(0,0)} = p$ and $y_{2(0,0)} = q$

then $\rho_{(0,0)} = \dfrac{(1+p^2)^{3/2}}{q}$ —— (A)

To find the values of p and q we expand $y=f(x)$ throug maclaurins

$\Rightarrow y = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \dots$

$\Rightarrow y = 0 + x.p + \dfrac{x^2}{2!} q + \dots$

$\Rightarrow y = px + \dfrac{qx^2}{2}$ + higher powers of x —— (1)

$\because p = 0$ then (1) implies —— $\left\{\because x=0, y=0 \Rightarrow y'=0\right\}$

$y = \dfrac{qx^2}{2}$ + higher powers of x

$\Rightarrow \dfrac{2y}{x^2} = q$ + higher power's of x

$\Rightarrow \lim\limits_{x \to 0} \dfrac{2y}{x^2} = q \Rightarrow \rho = \dfrac{(1+0)^{3/2}}{q}$

$$\Rightarrow \rho = \frac{1}{q}$$

$$\Rightarrow \boxed{\rho_{(0,0)} = \underset{x\to 0}{Lt} \frac{x^2}{2y}}$$

Also if the curve passes through origin and y-axis is tangent then

$$\boxed{\rho_{(0,0)} = \underset{y\to 0}{Lt} \frac{y^2}{2x}}$$

Curvature at Origin in polar form

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If the given equation of curve is in polar form and x-axis is tangent at origin then,

$$x = r\cos\theta, \; y = r\sin\theta$$

$$\because \rho_{(0,0)} = \underset{\cancel{x}\to 0}{Lt} \frac{x^2}{2y}$$

$$= \underset{\theta\to 0}{Lt} \frac{r^{\cancel{2}}\cos^2\theta}{2\cancel{r}\sin\theta}$$

$\because$ $x = r\cos\theta$ & $x \to 0$ $\Rightarrow \cos\theta \to 0 \Rightarrow \theta \to 0$

$$= \underset{\theta\to 0}{Lt} \frac{r}{2}\frac{\cos^2\theta}{\sin\theta}$$

$$= \underset{\theta\to 0}{Lt} \frac{r}{2\theta}\left(\frac{\theta}{\sin\theta}\right)\cos^2\theta$$

$\Rightarrow \rho = \lim_{\theta \to 0} \frac{r}{2\theta} \cdot (1) \cdot (1)$ $\quad \because \lim_{\theta \to 0} \cos\theta = 1$, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

$$\Rightarrow \boxed{\rho = \lim_{\theta \to 0} \frac{r}{2\theta}}$$

Ques Show that the radii of curvature of the curve $y^2 = x^2 \frac{(a+x)}{a-x}$ at the origin are $\pm\sqrt{2}\,a$

Soln $\because$ $y^2 = x^2 \frac{(a+x)}{(a-x)}$

$\Rightarrow y^2(a-x) = x^2(a+x)$ —— (1)

$\because$ $y = f(x)$ and $x=0$ at origin

$\therefore y = px + q\frac{x^2}{2} + \dots$

$\therefore$ (1) $\Rightarrow (a-x)\{px + q\frac{x^2}{2} + \dots\}^2 = x^2(a+x)$

$\Rightarrow (a-x)\{p^2x^2 + 2p \cdot q\frac{x^3}{2} + \dots\} = ax^2 + x^3$

Compairing co-efficients of x^2 and x^3 we get

$ap^2 = a \Rightarrow p^2 = 1 \Rightarrow p = \pm 1$

and $apq - p^2 = 1$

$\Rightarrow a \cdot p \cdot q - 1 = 1$

$\Rightarrow apq = 1+1 = 2 \Rightarrow q = \frac{2}{ap}$

If $p = 1 \Rightarrow q = \frac{2}{a}$, If $p = -1 \Rightarrow q = -2/a$

$$\therefore \rho = \frac{(1+(\pm 1)^2)^{3/2}}{q}$$

$$= \frac{(1+1)^{3/2}}{\pm(2/a)}$$

$$\Rightarrow \rho = \pm\frac{(2)^{3/2}}{2} \times a$$

$$\Rightarrow \rho = \pm(2^{3/2-1}) \times a$$

$$\Rightarrow \rho = \pm(2^{1/2})a$$

$$\Rightarrow \rho = \pm\sqrt{2}\, a$$

Proved

Ques 4 Show that the radii of curvature at the origin for the curve $x^3+y^3=3axy$ are each equal to $(3/2)a$.

Soln Given equation $x^3+y^3=3axy$ — (1) is of the curve passing through origin. So either ~~the is~~ x axis or y-axis is tangent.

If x-axis is tangent then we know that

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Now Equation ① implies

$$\frac{x^3}{2xy}+\frac{y^3}{2xy}=\frac{3axy}{2xy}$$

$$\Rightarrow \frac{x^2}{2y}+\frac{y^2}{2x}=\frac{3a}{2} \qquad \text{———} \quad (2)$$

$$\Rightarrow \frac{x^2}{2y}+\frac{1}{4}xy\cdot\frac{2y}{x^2}=\frac{3a}{2}$$

$$\Rightarrow \lim_{x,y\to 0}\left\{\frac{x^2}{2y}+\frac{1}{4}xy\cdot\frac{2y}{x^2}\right\}=\frac{3a}{2}$$

$$\Rightarrow \rho+\frac{1}{4}\cdot 0\cdot 0=\frac{3a}{2}$$

$$\Rightarrow \rho=\frac{3a}{2}$$

Similarly when y-axis is tangent

$$\cancel{\rho \neq \frac{y^2}{2x}} \quad \rho=\lim_{x,y\to 0}\frac{y^2}{2x}$$

Again from eqn ② we have

$$\left\{\frac{1}{4}\cdot xy\cdot\frac{x}{y^2}+\frac{y^2}{2x}\right\}=\frac{3a}{2}$$

$$\Rightarrow \lim_{x,y\to 0}\left\{\frac{1}{4}(x\cdot y)\cdot\frac{x}{y^2}+\frac{y^2}{2x}\right\}=\frac{3a}{2}$$

$$\Rightarrow \frac{1}{4}\cdot 0\cdot 0+\rho=\frac{3a}{2}$$

$$\Rightarrow \rho=\frac{3a}{2}$$

Proved